Linear Algebra

Comprehensive Review

A Connected Journey Through Linear Algebra

From Linear Systems to Spectral Theory

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1 Unifying Themes: The Three Fundamental Perspectives

Linear algebra is not just a collection of rules; it is a language spoken in three dialects. Mastery comes from the ability to translate fluidly between them.

The Three Interconnected Lenses

1. **The Computational Lens (The Matrix as Data)**
   Path: Systems of equations → Row reduction → Matrix factorization (LU, QR)
   - *Core Question:* "How do we compute the solution efficiently?"
   - *Key Insight:* Row operations change the matrix but preserve the **solution set** (Null Space) and linear dependencies among columns. We manipulate the data to reveal its "rank"—the true amount of information it contains.

2. **The Geometric Lens (The Matrix as Transformation)**
   Path: Vectors → Linear Transformations → Orthogonality → Least Squares
   - *Core Question:* "How does this matrix distort space?"
   - *Key Insight:* Every matrix A is a function $T(\mathbf{x}) = A\mathbf{x}$. It takes a grid and stretches, rotates, or collapses it. The **Determinant** measures the change in volume, and **Eigenvectors** reveal the axes that resist rotation.

3. **The Abstract Lens (The Matrix as Operator)**
   Path: Vector spaces → Basis independence → Spectral Theory
   - *Core Question:* "What are the intrinsic properties independent of coordinates?"
   - *Key Insight:* A "vector" is any object that obeys superposition (arrows, functions, polynomials). Coordinates are just a temporary lens; **Dimension** and **Eigenvalues** are the invariant truths of the space itself, regardless of the basis we choose.

Deep Insight: The Static vs. Dynamic Duality

One of the most profound realizations in Linear Algebra is that **solving equations (Static)** and **transforming space (Dynamic)** are the same problem:

- **Static View:** $A\mathbf{x} = \mathbf{b}$ asks, "Does the vector $\mathbf{b}$ lie in the subspace spanned by columns of A?"
- **Dynamic View:** $A\mathbf{x} = \mathbf{b}$ asks, "Is there an input $\mathbf{x}$ that gets mapped to $\mathbf{b}$ by the transformation A?"

The **Rank-Nullity Theorem** ties these worlds together: the dimension of the image (Rank) plus the dimension of the collapse (Nullity) equals the total dimension of the domain.

The Central Story: From Arithmetic to Structure

We begin by solving linear systems (Ch 1) using raw arithmetic. We then realize that matrices are not just tables of numbers, but algebraic objects with structure (Ch 2-3). We abstract these properties to general vector spaces (Ch 4) to apply our tools to functions and signals.
The turning point is **Eigenvalues (Ch 5)**, where we stop asking "where does $\mathbf{x}$ go?" and start asking "which vectors stay on their own line?". This leads to **Orthogonality (Ch 6)**, the geometry of "best approximations."
Finally, these threads unite in the **Spectral Theorem (Ch 7)**: Symmetric matrices (the most common in physics/data) can be decomposed into pure scaling along perpendicular axes—the perfect marriage of algebra and geometry.

2 Fundamental Subspaces and Their Relationships

2.1 The Four Fundamental Subspaces

For an $m \times n$ matrix A:

The Four Fundamental Subspaces

1. **Column Space**: $\text{Col}(A) \subseteq \mathbb{R}^m$
   - $\text{Col}(A) = \{\mathbf{b} : A\mathbf{x} = \mathbf{b} \text{ has a solution}\}$
   - Span of columns of A
   - Image/Range of the linear transformation
   - $\dim(\text{Col}\, A) = \text{rank}(A) = r$
   - **Basis**: Pivot columns of A
2. **Null Space**: $\text{Nul}(A) \subseteq \mathbb{R}^n$
   - $\text{Nul}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$
   - Set of all solutions to homogeneous system
   - Kernel of the transformation
   - $\dim(\text{Nul}\, A) = n - r$ (nullity)
   - **Basis**: Special solutions from free variables
3. **Row Space**: $\text{Row}(A) = \text{Col}(A^T) \subseteq \mathbb{R}^n$
   - Span of rows of A
   - $\dim(\text{Row}\, A) = \text{rank}(A) = r$
   - **Basis**: Nonzero rows of $\text{RREF}(A)$
   - $\text{Row}(A) = \text{Row}(\text{RREF}(A))$
4. **Left Null Space**: $\text{Nul}(A^T) \subseteq \mathbb{R}^m$
   - $\text{Nul}(A^T) = \{\mathbf{y} : A^T\mathbf{y} = \mathbf{0}\}$
   - $\dim(\text{Nul}\, A^T) = m - r$

The Rank-Nullity Theorem (Fundamental Theorem Part 1)

For an $m \times n$ matrix A:

$$\operatorname{rank}(A) + \dim(\operatorname{Nul} A) = n \quad \text{(number of columns)}$$

More generally:

$$\dim(\operatorname{Col} A) + \dim(\operatorname{Nul} A) = n$$

Orthogonality Relationships

- $\operatorname{Row}(A) \perp \operatorname{Nul}(A)$ in $\mathbb{R}^n$
- $\operatorname{Col}(A) \perp \operatorname{Nul}(A^T)$ in $\mathbb{R}^m$
- $\mathbb{R}^n = \operatorname{Row}(A) \oplus \operatorname{Nul}(A)$ (orthogonal direct sum)
- $\mathbb{R}^m = \operatorname{Col}(A) \oplus \operatorname{Nul}(A^T)$ (orthogonal direct sum)

2.2 Row Reduction Techniques

Computing Bases for Fundamental Subspaces:

Computational Procedure

For Column Space:

1. Row reduce A to REF or RREF
2. Identify pivot columns (columns with leading 1's)
3. Basis for $\operatorname{Col}(A)$ = corresponding columns from *original* A

For Null Space:

1. Row reduce $[A]$ to RREF
2. Write system in parametric form
3. Each free variable gives one basis vector

For Row Space:

1. Row reduce A to RREF
2. Basis = nonzero rows of $\operatorname{RREF}(A)$
3. Alternative: Transpose and find column space of A^T

The four fundamental subspaces completely characterize a matrix. Rank reveals how much "information" the matrix contains, while nullity measures how much "redundancy" exists.

3 Bases and Coordinate Systems

3.1 Basis Properties

Basis Characterizations

A set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for vector space V if and only if:

- $\mathcal{B}$ is linearly independent AND spans V
- $\mathcal{B}$ is a minimal spanning set
- $\mathcal{B}$ is a maximal linearly independent set
- Every vector in V can be written *uniquely* as a linear combination of vectors in $\mathcal{B}$

Important Facts About Bases:

- If $\dim(V) = n$:
  - Any linearly independent set of n vectors is a basis
  - Any spanning set of n vectors is a basis
  - Any linearly independent set has $\leq n$ vectors
  - Any spanning set has $\geq n$ vectors

3.2 Coordinate Vectors

Definition: If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the **coordinate vector** of $\mathbf{x}$ relative to $\mathcal{B}$.

Computing Coordinate Vectors:

- Form the equation: $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$
- Write as matrix equation: $[\mathbf{b}_1 \cdots \mathbf{b}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{x}$
- Solve: $[\mathbf{x}]_{\mathcal{B}} = [\mathbf{b}_1 \cdots \mathbf{b}_n]^{-1}\mathbf{x}$

3.3 Change of Basis

Change of Basis Formula

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for V.
Change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Then:

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}}$$

Computing Change-of-Basis Matrix:

1. Find $[\mathbf{b}_i]_{\mathcal{C}}$ for each $\mathbf{b}_i$ in $\mathcal{B}$
2. Form matrix with these coordinate vectors as columns
3. Alternative: Row reduce $[\mathbf{c}_1 \cdots \mathbf{c}_n \mid \mathbf{b}_1 \cdots \mathbf{b}_n]$ to $[I \mid P]$

Properties:

- $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$
- Change of basis matrices are always invertible

Different bases provide different "coordinate systems" for the same space. Choosing the right basis can dramatically simplify computations (e.g., eigenvector bases for diagonalization).

4 Eigenvalues and Eigenvectors: Deep Dive

4.1 Fundamental Definitions and Computations

Definition: A nonzero vector $\mathbf{v}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}$$

Eigenspace: $E_\lambda = \text{Nul}(A - \lambda I)$ = subspace of all eigenvectors for λ (plus $\mathbf{0}$)

Computing Eigenvalues and Eigenvectors

Complete Step-by-Step Procedure:
Step 1: Find Eigenvalues

1. Form the characteristic equation: $\det(A - \lambda I) = 0$
2. Expand the determinant to get characteristic polynomial $p(\lambda)$
3. Solve for λ (roots of the polynomial)

Step 2: Find Eigenvectors

1. For each eigenvalue λ:
2. Form the matrix $(A - \lambda I)$
3. Solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$
4. Write solution in parametric vector form
5. Basis vectors for the parametric solution are eigenvectors

Step 3: Verify

1. Check: $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenpair
2. Verify: $\det(A - \lambda I) = 0$ for each eigenvalue

4.2 Special Cases and Shortcuts

Fast Eigenvalue Computation

Triangular Matrices:

- For upper or lower triangular A: eigenvalues = diagonal entries
- Example: $A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalues $\lambda = 2, -3, 1$

Diagonal Matrices:

- $A = \text{diag}(d_1, d_2, \ldots, d_n)$ has eigenvalues $d_1, d_2, \ldots, d_n$
- Eigenvectors are standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$

2×2 **Matrices**:

- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
$$\det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$
where $\text{tr}(A) = a + d$
- Eigenvalues: $\lambda = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}(A))^2 - 4\det(A)}}{2}$

Block Diagonal Matrices:

- If $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, then eigenvalues of A = eigenvalues of $B \cup$ eigenvalues of C

4.3 Properties of Eigenvalues

Important Eigenvalue Properties

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicities):

1. **Trace**: $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
2. **Determinant**: $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$
3. **Powers**: If λ is eigenvalue of A, then λ^k is eigenvalue of A^k
4. **Polynomials**: If $p(\lambda) = a_0 + a_1\lambda + \cdots + a_k\lambda^k$, then eigenvalues of $p(A)$ are $p(\lambda_1), \ldots, p(\lambda_n)$
5. **Inverse**: If A is invertible with eigenvalue λ, then A^{-1} has eigenvalue $\frac{1}{\lambda}$
6. **Transpose**: A and A^T have the same eigenvalues
7. **Similarity**: Similar matrices $B = P^{-1}AP$ have identical eigenvalues

4.4 Characteristic Polynomials

Characteristic Polynomial: $p(\lambda) = \det(A - \lambda I)$

- Degree n polynomial in λ
- Roots of $p(\lambda)$ are the eigenvalues

Algebraic vs Geometric Multiplicity:

- **Algebraic multiplicity** of λ: multiplicity as root of characteristic polynomial
- **Geometric multiplicity** of λ: $\dim(E_\lambda) = \dim(\text{Nul}(A - \lambda I))$
- Always: geometric multiplicity $\leq$ algebraic multiplicity
- Equality holds $\Leftrightarrow$ eigenspace has "enough" eigenvectors

4.5 Constructing Matrices from Eigendata

Building Matrices from Eigenvalues and Eigenvectors

Given: Eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$
Method 1: Diagonalization Formula

$$A = PDP^{-1}$$

where $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ and $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Method 2: Spectral Decomposition (for symmetric A with orthonormal eigenvectors)

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

Procedure:

1. Arrange eigenvectors as columns of P
2. Create diagonal matrix D with eigenvalues
3. Compute $A = PDP^{-1}$ (or $A = PDP^T$ if eigenvectors are orthonormal)
4. Verify: Check that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$

Eigenvectors reveal the "natural directions" of a matrix transformation. Triangular matrices make eigenvalues immediately visible on the diagonal. The relationship between A and A^TA connects to the singular value decomposition.

5 Matrix Diagonalization

5.1 Diagonalization Theory

Definition: An $n \times n$ matrix A is **diagonalizable** if

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
Construction:

- $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ where $\mathbf{v}_i$ are eigenvectors
- $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ where λ_i are corresponding eigenvalues
- Then: $AP = PD$ and $A = PDP^{-1}$

5.2 Conditions for Diagonalizability

When is A Diagonalizable?

Sufficient Conditions:

1. A has n *distinct* eigenvalues
   - Eigenvectors from different eigenvalues are automatically linearly independent
2. For each eigenvalue λ: geometric multiplicity = algebraic multiplicity
   - i.e., $\dim(E_\lambda)$ = multiplicity of λ in characteristic polynomial
3. A is symmetric
   - Symmetric matrices are *always* diagonalizable (Spectral Theorem)
   - Moreover, they are *orthogonally* diagonalizable

When A is NOT Diagonalizable:

- Insufficient eigenvectors (geometric multiplicity $<$ algebraic multiplicity)
- Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigenvalue $\lambda = 1$ but $\dim(E_1) = 1 < 2$

5.3 Computing Powers via Diagonalization

Computing A^k Efficiently

Problem: Compute A^k for large k
Solution: If $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}$$

where $D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$ (easy to compute!)

Procedure:

1. Diagonalize: Find P and D such that $A = PDP^{-1}$
2. Compute diagonal power: $D^k = \text{diag}(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$
3. Reconstruct: $A^k = PD^kP^{-1}$

Applications:

- Solving difference equations: $\mathbf{x}_{k+1} = A\mathbf{x}_k \Rightarrow \mathbf{x}_k = A^k\mathbf{x}_0$
- Matrix exponential: $e^{At} = Pe^{Dt}P^{-1}$

5.4 Orthogonal Diagonalization

The Spectral Theorem (Orthogonal Diagonalization)

An $n \times n$ matrix A is **orthogonally diagonalizable** (i.e., $A = PDP^T$ with P orthogonal) if and only if A is symmetric.
Spectral Decomposition:

$$A = PDP^T = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$$

where:

- P is orthogonal matrix: $P^TP = I$ and $P^T = P^{-1}$
- Columns of P are orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$
- D is diagonal with eigenvalues on diagonal
- All eigenvalues of symmetric matrices are *real*
- Eigenvectors from different eigenspaces are *orthogonal*

Why Orthogonal Diagonalization Matters:

- $P^{-1} = P^T$ is easy to compute (just transpose!)
- Orthogonal matrices preserve lengths and angles
- Numerical stability in computations
- Foundation for principal component analysis (PCA)

Diagonalization transforms A into its simplest form. For symmetric matrices, we get the bonus of orthogonal eigenvectors, making P^{-1} trivial to compute. This is the foundation of spectral theory.

6 Orthogonality: Theory and Computations

6.1 Inner Products and Norms

Inner Product (Dot Product): For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties:

- Symmetry: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- Linearity: $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- Homogeneity: $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- Positive definiteness: $\mathbf{u} \cdot \mathbf{u} \geq 0$, with equality iff $\mathbf{u} = \mathbf{0}$

Norm (Length): $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
Distance: $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$
Unit Vector: $\|\mathbf{u}\| = 1$. Normalize: $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$

6.2 Orthogonality

Definition: Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. Write: $\mathbf{u} \perp \mathbf{v}$.
Pythagorean Theorem: If $\mathbf{u} \perp \mathbf{v}$, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

6.3 Orthogonal Sets and Bases

Orthogonal and Orthonormal Sets

Orthogonal Set: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ where $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$
Orthonormal Set: Orthogonal set where each $\|\mathbf{u}_i\| = 1$
Key Theorem: An orthogonal set of *nonzero* vectors is linearly independent.

Coordinates in Orthogonal Basis:
If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for subspace W, then for any $\mathbf{y} \in W$:

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

For orthonormal basis (simpler!):

$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$

6.4 Orthogonal Matrices

Orthogonal Matrix Properties

A square matrix U is **orthogonal** if its columns form an orthonormal set.
Equivalent Conditions:

- $U^TU = I$
- $UU^T = I$
- $U^T = U^{-1}$
- Columns of U are orthonormal
- Rows of U are orthonormal

Properties of Orthogonal Matrices:

1. **Preserve lengths**: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$
2. **Preserve angles**: $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. **Preserve distances**: $\|U\mathbf{x} - U\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$
4. $\det(U) = \pm 1$
5. Product of orthogonal matrices is orthogonal
6. Inverse of orthogonal matrix is orthogonal

6.5 Orthogonal Projections

Projection onto a Vector:
For nonzero vector $\mathbf{u}$, the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is:

$$\text{proj}_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$$

Projection onto a Subspace:
If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for subspace W:

$$\text{proj}_W\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

For orthonormal basis:

$$\text{proj}_W\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$

In matrix form: If $U = [\mathbf{u}_1 \cdots \mathbf{u}_p]$ has orthonormal columns:

$$\text{proj}_W\mathbf{y} = UU^T\mathbf{y}$$

Orthogonal Decomposition:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} = \text{proj}_W\mathbf{y} \in W$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} \perp W$

Best Approximation Theorem: $\hat{\mathbf{y}} = \text{proj}_W\mathbf{y}$ is the closest point in W to $\mathbf{y}$:

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \text{for all } \mathbf{v} \in W, \mathbf{v} \neq \hat{\mathbf{y}}$$

6.6 Gram-Schmidt Orthogonalization

The Gram-Schmidt Process

Input: Basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for subspace W
Output: Orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ for W
Algorithm:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{i=1}^{p-1} \frac{\mathbf{x}_p \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\mathbf{v}_i
\end{aligned}$$

To get orthonormal basis: Normalize each $\mathbf{v}_i$:

$$\mathbf{u}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$$

QR Factorization:
If $A = [\mathbf{x}_1 \cdots \mathbf{x}_n]$ has linearly independent columns, then:

$$A = QR$$

where:

- Q has orthonormal columns (from Gram-Schmidt)
- R is upper triangular with positive diagonal entries

6.7 How Orthogonality Simplifies Computations

Computational Advantages of Orthogonality

1. **Finding coordinates**: No need to solve systems!
   - Orthogonal basis: $c_i = \frac{\mathbf{x} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$
   - Orthonormal basis: $c_i = \mathbf{x} \cdot \mathbf{u}_i$ (just dot products!)
2. **Matrix inversion**: For orthogonal U:
   - $U^{-1} = U^T$ (transpose instead of Gaussian elimination!)
3. **Projections**: Simple formulas
   - $\text{proj}_W \mathbf{y} = UU^T\mathbf{y}$ for orthonormal basis
4. **Least-squares**: QR factorization dramatically simplifies computation
   - **Master the standard approach**: $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$
   - **QR approach**: With $A = QR$, we get $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$
5. **Eigenvalue problems**: Symmetric matrices
   - Guaranteed real eigenvalues
   - Orthogonal eigenvectors
   - $A = PDP^T$ with $P^{-1} = P^T$

6.8 Orthogonal Eigenvectors of Symmetric Matrices

Properties of Symmetric Matrices

If A is symmetric ($A = A^T$):

1. All eigenvalues are **real**
2. Eigenvectors corresponding to **distinct** eigenvalues are **orthogonal**
3. A is **orthogonally diagonalizable**: $A = PDP^T$ where P is orthogonal
4. Can always find an orthonormal basis of eigenvectors for $\mathbb{R}^n$

Constructing Orthonormal Eigenvectors:

1. Find eigenvalues and eigenvectors normally
2. For each eigenspace E_λ:
   - If $\dim(E_\lambda) = 1$: Just normalize the eigenvector
   - If $\dim(E_\lambda) > 1$: Apply Gram-Schmidt to basis of E_λ
3. Eigenvectors from different eigenspaces are already orthogonal (no work needed!)
4. Result: Orthonormal basis of eigenvectors

Orthogonality is the "golden key" that unlocks computational simplicity. Orthogonal bases eliminate the need to solve systems, orthogonal matrices have trivial inverses, and symmetric matrices guarantee orthogonal eigenvectors.

7 Quadratic Forms

7.1 Definition and Matrix Representation

Quadratic Form: A function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

where A is a symmetric $n \times n$ matrix.

Expanded Form:

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

Example in $\mathbb{R}^2$

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note: Coefficient of x_1x_2 is split equally between a_{12} and a_{21} to maintain symmetry.

Finding Matrix from Quadratic Form:

- Diagonal entries: coefficients of x_i^2 terms
- Off-diagonal entries: $a_{ij} = a_{ji} = \frac{1}{2}$(coefficient of x_ix_j term)

7.2 Principal Axes Theorem

Principal Axes Theorem

Let A be a symmetric $n \times n$ matrix. Then there exists an orthogonal change of variable

$$\mathbf{x} = P\mathbf{y}$$

where P orthogonally diagonalizes A ($A = PDP^T$), that transforms the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ into

$$Q(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.
The columns of P are the **principal axes** (eigenvectors of A).

Proof Sketch:

$$\begin{aligned} Q(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} \\ &= (P\mathbf{y})^T A(P\mathbf{y}) \\ &= \mathbf{y}^T P^T AP\mathbf{y} \\ &= \mathbf{y}^T D\mathbf{y} \quad (\text{since } P^T AP = D) \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

7.3 Change of Variables to Eliminate Cross-Product Terms

Eliminating Cross-Product Terms

Problem: Given $Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, eliminate the x_1x_2 term.
Procedure:

1. Write as matrix form: $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
2. Find eigenvalues of A: Solve $\det(A - \lambda I) = 0$
3. Find orthonormal eigenvectors: These form columns of P
4. Change variables: $\mathbf{x} = P\mathbf{y}$
5. Result: $Q(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2$ (no cross-product term!)

Geometric Interpretation:

- Original coordinates (x_1, x_2) may be "tilted"
- New coordinates (y_1, y_2) align with principal axes (eigenvectors)
- The quadratic form describes an ellipse, hyperbola, or parabola
- Rotation to principal axes reveals the shape clearly

7.4 Classifying Definiteness

Definiteness Classification

Let A be symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$.

1. **Positive Definite**: $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$
   - $\Leftrightarrow$ All eigenvalues > 0
2. **Positive Semidefinite**: $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$
   - $\Leftrightarrow$ All eigenvalues ≥ 0
3. **Negative Definite**: $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$
   - $\Leftrightarrow$ All eigenvalues < 0
4. **Negative Semidefinite**: $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$
   - $\Leftrightarrow$ All eigenvalues ≤ 0
5. **Indefinite**: Q takes both positive and negative values
   - $\Leftrightarrow$ Eigenvalues have mixed signs

Quick Tests:

- 2×2 **case**: For $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$:
  - Positive definite $\Leftrightarrow a > 0$ and $\det(A) = ac - b^2 > 0$

- Negative definite $\Leftrightarrow a < 0$ and $\det(A) > 0$
- Indefinite $\Leftrightarrow \det(A) < 0$

Quadratic forms describe geometric objects (ellipsoids, hyperboloids, paraboloids). Eigenvalues determine the "shape" - whether it opens up, down, or saddles. Principal axes align with eigenvectors, eliminating cross-product terms.

8 Least-Squares Problems

8.1 The Least-Squares Problem

Problem Statement: Given $A\mathbf{x} = \mathbf{b}$ where the system is *inconsistent* (no exact solution), find $\hat{\mathbf{x}}$ that minimizes

$$\|A\mathbf{x} - \mathbf{b}\|$$

This $\hat{\mathbf{x}}$ is called the **least-squares solution**.
Geometric Interpretation:

- $\mathbf{b}$ is not in $\text{Col}(A)$
- Find the closest point in $\text{Col}(A)$ to $\mathbf{b}$
- This closest point is $\text{proj}_{\text{Col}(A)}\mathbf{b} = A\hat{\mathbf{x}}$
- The error vector $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to $\text{Col}(A)$

8.2 Normal Equations

The Normal Equations

The least-squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ satisfies

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

This is always consistent (has at least one solution).

8.3 Computing Least-Squares Solutions

Procedure for Least-Squares

Method 1: Normal Equations (at least master this standard method)

1. Compute A^TA and $A^T\mathbf{b}$
2. Solve $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ (use row reduction or matrix inverse)
3. If A^TA is invertible: $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$

Method 2: QR Factorization (more numerically stable)

1. Factor $A = QR$ where Q has orthonormal columns
2. Then: $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$
3. Solve $R\hat{\mathbf{x}} = Q^T\mathbf{b}$ by back-substitution

8.4 Uniqueness Conditions

When is Least-Squares Solution Unique?

The least-squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ is unique if and only if:

- Columns of A are linearly independent
- $\Leftrightarrow \text{Nul}(A) = \{\mathbf{0}\}$
- $\Leftrightarrow A^TA$ is invertible
- $\Leftrightarrow \text{rank}(A) = n$ (number of columns)

If columns are linearly independent:

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$$

The matrix $(A^TA)^{-1}A^T$ is called the **pseudoinverse** of A.

9 Spectral Theory and Applications

9.1 Spectral Decomposition

Spectral Decomposition of Symmetric Matrices

If A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then:
Form 1: Matrix Factorization

$$A = PDP^T$$

where $P = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]$ is orthogonal and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$
Form 2: Outer Product Expansion

$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$$

Each $\mathbf{u}_i\mathbf{u}_i^T$ is a rank-1 projection matrix onto the eigenspace E_{λ_i}.

Interpretation:

- A acts as scaled projection along each eigendirection
- Eigenvalue λ_i determines scaling factor in direction $\mathbf{u}_i$
- This is the "atomic decomposition" of A

9.2 Applications of Eigenvalue Decomposition

Key Applications

1. **Matrix Powers**: $A^k = PD^kP^T$ (easy when D is diagonal)
2. **Matrix Exponential**: $e^{At} = Pe^{Dt}P^T$ where $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$
3. **Differential Equations**: Solve $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ using $\mathbf{x}(t) = e^{At}\mathbf{x}_0$
4. **Quadratic Forms**: Classify via eigenvalues (positive definite, etc.)
5. **Principal Component Analysis (PCA)**:
   - Eigenvectors = principal components (directions of maximum variance)
   - Eigenvalues = variance along each component
6. **Vibration Analysis**: Natural frequencies from eigenvalues
7. **Stability Analysis**: System stable if all eigenvalues have negative real parts
8. **PageRank Algorithm**: Largest eigenvalue and eigenvector of transition matrix
9. **Quantum Mechanics**: Observable operators, ground state energy
10. **Optimization**: the eigenvalues of Hessian matrix

9.3 The Fundamental Theorem of Linear Algebra

The Fundamental Theorem of Linear Algebra

For an $m \times n$ matrix A with rank r:
Dimensions:

$$\begin{aligned}\dim(\text{Col}\, A) &= r \\ \dim(\text{Nul}\, A) &= n - r \\ \dim(\text{Row}\, A) &= r \\ \dim(\text{Nul}\, A^T) &= m - r\end{aligned}$$

Orthogonality:

$$\begin{aligned}\text{Row}(A) \perp \text{Nul}(A) \quad &\text{in } \mathbb{R}^n \\ \text{Col}(A) \perp \text{Nul}(A^T) \quad &\text{in } \mathbb{R}^m\end{aligned}$$

Direct Sums:

$$\begin{aligned}\mathbb{R}^n &= \text{Row}(A) \oplus \text{Nul}(A) \\ \mathbb{R}^m &= \text{Col}(A) \oplus \text{Nul}(A^T)\end{aligned}$$

Isomorphisms:

- $A : \text{Row}(A) \to \text{Col}(A)$ is an isomorphism (one-to-one and onto)
- $A^T : \text{Col}(A) \to \text{Row}(A)$ is an isomorphism

Visualization: The "Big Picture" of Linear Algebra
The diagram is:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \\ \| & & \| \\ \mathrm{Row}(A) \oplus \mathrm{Nul}(A) & \xrightarrow{A} & \mathrm{Col}(A) \oplus \mathrm{Nul}(A^T) \\ \underbrace{\dim r \quad + \quad \dim(n-r)}_{\text{both in } \mathbb{R}^n} & & \underbrace{\dim r \quad + \quad \dim(m-r)}_{\text{both in } \mathbb{R}^m} \\ \mathrm{Row}(A) \perp \mathrm{Nul}(A) & & \mathrm{Col}(A) \perp \mathrm{Nul}(A^T) \end{array}$$

The Fundamental Theorem synthesizes everything: it connects dimensions (rank-nullity), geometry (orthogonality and direct sums), and transformations (isomorphism from $\mathrm{Row}(A)$ to $\mathrm{Col}(A)$). This is the complete picture of what a matrix does - it reveals both where A acts as an isomorphism and where it annihilates information!

10 The Big Picture: How Everything Connects

10.1 The Central Thread

1. **Foundation**: Solving $A\mathbf{x} = \mathbf{b}$ (Chapters 1-2)
   - Row reduction, matrix operations, invertibility
   - Foundation: understanding when and how systems have solutions
2. **Structure**: Understanding subspaces and dimension (Chapters 2-4)
   - Column space, null space, row space, bases, dimension, rank
   - The four fundamental subspaces completely characterize A
3. **Transformation**: Eigenvalues reveal intrinsic properties (Chapter 5)
   - Eigenvectors, diagonalization, similarity
   - Natural directions where A acts as pure scaling
4. **Geometry**: Orthogonality provides optimal coordinates (Chapter 6)
   - Inner products, projections, Gram-Schmidt, least squares
   - Orthogonal bases make everything simpler
5. **Synthesis**: Spectral theorem unifies everything for symmetric matrices (Chapter 7)
   - Orthogonal diagonalization, quadratic forms, definiteness
   - Perfect synthesis: $A = PDP^T$ with P orthogonal

10.2 Three Essential Theorems

The Three Pillars of Linear Algebra

1. **Invertible Matrix Theorem (Ch 2.3)**: The master list of equivalences
   - Connects invertibility, linear independence, span, transformations, determinant, eigenvalues
   - many equivalent conditions
2. **Rank-Nullity Theorem (Ch 2.9, 4.5)**: Fundamental dimension constraint

$$\text{rank}(A) + \dim(\text{Nul}\, A) = n$$

   - Relates the "information content" (rank) to "redundancy" (nullity)
3. **Spectral Theorem (Ch 7.1)**: Ultimate result for symmetric matrices

$$A \text{ symmetric} \Rightarrow A = PDP^T \text{ with } P \text{ orthogonal}$$

   - Guarantees real eigenvalues and orthonormal eigenvectors
   - Foundation for quadratic forms, PCA, and countless applications

10.3 The Factorization Story

Factorization	Form	Purpose	Requirements
LU	$A = LU$	Computational efficiency	Square, no row swaps
QR	$A = QR$	Orthogonalization	Linearly indep. cols
Eigendecomposition	$A = PDP^{-1}$	Diagonalization	n lin. indep. eigenvectors
Spectral	$A = PDP^T$	Orthogonal diag.	Symmetric matrix

Each factorization reveals different structure:

- **LU**: Efficient for solving multiple systems with same A
- **QR**: Numerically stable, good for least-squares
- **Eigendecomposition**: Powers, differential equations, dynamics
- **Spectral**: Quadratic forms, PCA, optimization

10.4 Key Conceptual Bridges

Fundamental Equivalences

- $A\mathbf{x} = \mathbf{b}$ has solution $\Leftrightarrow$ $\mathbf{b} \in \text{Col}(A)$ $\Leftrightarrow$ $\mathbf{b}$ is linear combination of columns
- Linear independence $\Leftrightarrow$ $A\mathbf{x} = \mathbf{0}$ only for $\mathbf{x} = \mathbf{0}$ $\Leftrightarrow$ $\text{Nul}(A) = \{\mathbf{0}\}$
- Basis $\Leftrightarrow$ Minimal spanning set $\Leftrightarrow$ Maximal independent set $\Leftrightarrow$ Unique coordinates
- Invertibility $\Leftrightarrow$ n pivots $\Leftrightarrow$ $\det(A) \neq 0$ $\Leftrightarrow$ $\lambda = 0$ not an eigenvalue
- Eigenvalue λ $\Leftrightarrow$ $\det(A - \lambda I) = 0$ $\Leftrightarrow$ $\text{Nul}(A - \lambda I) \neq \{\mathbf{0}\}$
- Orthogonality $\Leftrightarrow$ $\mathbf{u} \cdot \mathbf{v} = 0$ $\Leftrightarrow$ Pythagorean theorem holds $\Leftrightarrow$ Independent + best approximation

10.5 Analyzing Linear Algebra Problems

The following are strategic tips for analyzing problems in linear algebra. These provide examples of strategies for checking your understanding and approaching problems systematically.

1. **Identify the question type**:
   - Existence? (Does a solution exist?) $\rightarrow$ Check if $\mathbf{b} \in \text{Col}(A)$
   - Uniqueness? (Is the solution unique?) $\rightarrow$ Check if $\text{Nul}(A) = \{\mathbf{0}\}$
   - Computation? (Find explicit solution) $\rightarrow$ Row reduction or inverse
   - Structure? (Describe solution set) $\rightarrow$ Parametric vector form
   - Best approximation? $\rightarrow$ Least-squares/projection
2. **Choose the right tool**:
   - Systems of equations $\rightarrow$ Row reduction to RREF
   - Invertibility $\rightarrow$ Check pivots, determinant, or eigenvalues
   - Subspaces $\rightarrow$ Find bases and check dimension
   - Transformation analysis $\rightarrow$ Eigenvalues and eigenvectors
   - Quadratic forms $\rightarrow$ Eigenvalues for classification
3. **Check dimensions**:
   - Does $\text{rank}(A) + \dim(\text{Nul}\, A) = n$? Be careful with $m \times n$ matrices
   - Do matrix dimensions match for multiplication?
   - Is the basis count equal to the dimension?
   - Verify rank, subspace dimensions, and number of eigenvalues
4. **Verify computations**:
   - Plug solutions back into original equations
   - Check $A\mathbf{v} = \lambda\mathbf{v}$ for eigenpairs
   - Verify orthogonality: $\mathbf{u} \cdot \mathbf{v} = 0$
   - Check special cases (triangular, diagonal, symmetric)
5. **Analyze matrix structure**:
   - Is the matrix invertible?
   - What are the rank and dimensions of subspaces related to the matrix?
   - Is the matrix symmetric?
   - Is the matrix orthogonal?
   - Can we use eigendecomposition or spectral decomposition?
   - Are two matrices similar?
6. **Interpret geometrically**:
   - What transformation does this represent?
   - What is the geometric meaning?
   - Does orthogonality simplify the picture?

11 The Final Insight

The Essence of Linear Algebra

Linear algebra studies **linear transformations** through three equivalent representations:

- **Geometric**: Transformations of space
  - Rotations, reflections, projections, scaling
  - Subspaces, orthogonality, distances
- **Algebraic**: Matrix multiplication
  - $T(\mathbf{x}) = A\mathbf{x}$
  - Eigenvalues, diagonalization, factorizations
- **Coordinate-based**: Systems of linear equations
  - $A\mathbf{x} = \mathbf{b}$
  - Row reduction, bases, coordinates

The power comes from moving fluidly between these perspectives, choosing whichever provides the clearest insight for the problem at hand. The **Spectral Theorem** for symmetric matrices represents the culmination: every symmetric transformation can be understood as *pure scaling along orthogonal axes*:

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

This is the perfect synthesis of:

- **Algebra**: Matrix factorization $A = PDP^T$
- **Geometry**: Orthogonal eigenvectors as coordinate axes
- **Analysis**: Eigenvalues as scaling factors

The Fourth Pillar: The Functional Perspective

Beyond these three views lies a deeper insight: *matrices are finite-dimensional shadows of infinite-dimensional operators.* Every concept in linear algebra extends to:

- **Function spaces**: $L^2[0, 1]$, spaces of continuous functions
- **Linear operators**: Differential operators $\frac{d}{dx}$, integral operators $\int_0^x f(t)\,dt$
- **Inner products**: $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$

This reveals that matrices are *discretizations* of continuous phenomena. The eigenvalue equation $A\mathbf{v} = \lambda\mathbf{v}$ becomes the differential equation $\mathcal{L}f = \lambda f$, connecting linear algebra to Fourier analysis, quantum mechanics, and PDEs.

Beyond the Fundamentals

- **Matrix Decompositions**: Beyond eigendecomposition
  - Singular Value Decomposition (SVD): $A = U\Sigma V^T$
  - QR, Cholesky, Schur decompositions
- **Matrix Functions and Calculus**
  - Matrix exponential e^{At} — solving differential equations $\dot{\mathbf{x}} = A\mathbf{x}$
  - Matrix logarithm, square roots, and general $f(A)$ via spectral calculus
  - Gradient descent on manifolds: $\frac{\partial}{\partial A}\text{tr}(AB)$
- **Linear Algebra over Arbitrary Fields**
  - From $\mathbb{R}$ and $\mathbb{C}$ to $\mathbb{F}_p$ (finite fields) — coding theory, cryptography
  - Jordan normal form
- **Representation Theory**
  - Groups acting via matrices
  - Applications: Particle physics, crystallography, quantum computing
- **Infinite-Dimensional Vector Spaces**
  - Hilbert spaces: ℓ^2, $L^2[0,1]$ — the infinite-dimensional $\mathbb{R}^n$
  - Banach spaces — when we drop the inner product
  - Spectral theory for unbounded operators
- **Matrix Manifolds and Optimization**
  - Optimization on $\text{SO}(n)$, Stiefel, Grassmann manifolds
  - Low-rank matrix completion and robust PCA
- **Quantum Information Theory**
  - Density matrices ρ — quantum states as positive semidefinite matrices
  - Unitary evolution $U\rho U^\dagger$
- **Machine Learning and Data Science**

The Unifying Thread: As you venture into these advanced topics, you'll find that the core principles remain constant. Whether studying infinite-dimensional operators, matrix manifolds, or quantum states, you're still asking:

What does this linear structure preserve?
How can I change coordinates to simplify?
What is the geometric meaning?

Linear algebra is not just a subject — it's a **way of thinking** that pervades modern mathematics, physics, information processing, and computation.